MATH 2230A

Tutorial 7 26-10-2016

Topics: ◦ Mid-term solution

◦ Contour integral of rational function

Mid-term solution

1) Note that (1) $|z-1| \le |z|+|1| = 4$

(2) $|z^2+5| \ge |z|^2-5 = 9-5 = 4$

(3) $|z^2+1| \ge |z|^2-1 = 9-1 = 8$.

So we have

$$\left|\frac{z-1}{z^4+6z^2+5}\right| = \frac{|z-1|}{|z^2+1||z^2+5|} \le \frac{4}{4\times 8} = \frac{1}{8}.$$

2) (Method I: comparing real and imaginary parts)

Recall that $\sin z = \sin x\cosh y + i\cos x\sinh y$

Hence $\sin z = \cosh 4$

$$\Longleftrightarrow \begin{cases} \sin x\cosh y = \cosh 4 & \text{—(1)} \\ \cos x \sinh y = 0 & \text{—(2)} \end{cases}$$

From (2), if $\sinh y = 0$, then $y=0$.

However if $y=0$, by (1), $\sin x = \cosh 4 > 1$, contradiction.

Hence $\cos x = 0$, $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

From (1), since $\cosh y$ and $\cosh 4$ are positive, $\sin x$ must be positive. So $x = \frac{\pi}{2} + 2n\pi$ and $\sin x = 1$.

As a result, $\cosh y = \cosh 4$, $y = \pm 4$, $z = (\frac{\pi}{2}+2n\pi) \pm 4i$

(Method II: solving the quadratic equation)

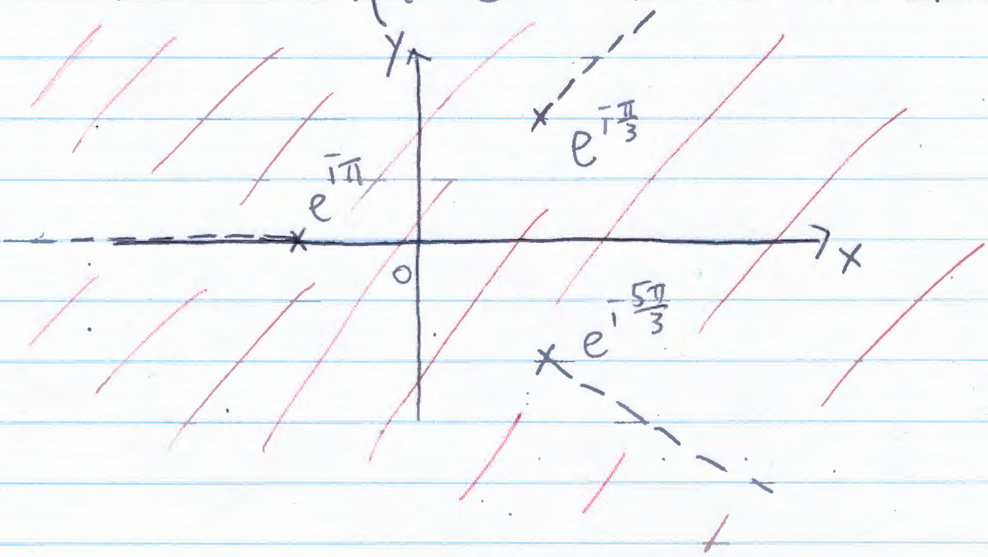
$$\sin z = \cosh 4 \Longleftrightarrow z = -i\log[i\cosh 4 + (1-\cosh^2 4)^{\frac{1}{2}}]$$
$$= -i\log[i\cosh 4 + (-\sinh^2 4)^{\frac{1}{2}}]$$
$$= -i\log[i(\cosh 4 \pm \sinh 4)]$$
$$= -i\log(ie^4) \quad \text{or} \quad -i\log(ie^{-4})$$

$$\Longleftrightarrow \quad z = -i\left[\ln|ie^{4}| + i\left(\tfrac{\pi}{2}+2k\pi\right)\right] \text{ or } z = -i\left[\ln|ie^{-4}| + i\left(\tfrac{\pi}{2}+2k\pi\right)\right]$$
$$= -i\left(4 + i\left(\tfrac{\pi}{2}+2k\pi\right)\right) \text{ or } z = -i\left(-4 + i\left(\tfrac{\pi}{2}+2k\pi\right)\right)$$
$$= \left(\tfrac{\pi}{2}+2k\pi\right) \pm 4i$$

3) Note that $1+z^3 = -r \quad , r \geq 0$

$$\Longleftrightarrow \quad z^3 = -(1+r)$$
$$\Longleftrightarrow \quad z^3 = (1+r)e^{i\pi}$$
$$\Longleftrightarrow \quad z = \sqrt[3]{1+r}\; e^{i\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)} \quad , k=0,1,2$$
$$\Longleftrightarrow \quad z = \tilde{r}e^{i\left(\frac{\pi}{3}+\frac{2k\pi}{3}\right)} \quad , \tilde{r} \geq 1$$

Since composition of analytic functions are analytic, the maximum domain is $\mathbb{C} \setminus \left(\{\tilde{r}e^{i\frac{\pi}{3}} \mid \tilde{r}\geq 1\} \cup \{\tilde{r}e^{i\pi} \mid \tilde{r}\geq 1\} \cup \{\tilde{r}e^{i\frac{5\pi}{3}} \mid \tilde{r}\geq 1\}\right)$



4) a) $\begin{cases} rU_r = V_\theta \\ U_\theta = -rV_r \end{cases}$

b) $u(r,\theta) = e^{-\theta}\cos(\ln r)$, $v(r,\theta) = e^{-\theta}\sin(\ln r)$.

Note that

$$\begin{cases} rU_r = r\left(-\frac{1}{r}e^{-\theta}\sin(\ln r)\right) = V_\theta \\ U_\theta = -e^{-\theta}\cos(\ln r) = -rV_r \end{cases}$$

Since $U_r, U_\theta, V_r, V_\theta$ exist, continuous on $(r>0, \theta\in(0,2\pi))$ and satisfy the CR-equation, $f(z)$ is differentiable over $(r>0, \theta\in(0,2\pi))$.

c) $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$= \frac{1}{e^{i\theta}}\left(\frac{-1}{r}e^{-\theta}\sin(\ln r) + i\,\frac{1}{r}e^{-\theta}\cos(\ln r)\right)$$

$$= \frac{i}{re^{i\theta}}\left(e^{-\theta}\cos(\ln r) + ie^{-\theta}\sin(\ln r)\right)$$

$$= \frac{i}{z}f(z)$$

5) See tutorial notes 1) #1.

Contour integral of rational function

(no 'hole')

Recall: (1) (Cauchy–Goursat theorem)

If f is analytic over a simply connected domain Ω, then $\int_C f\,dz = 0$ $\forall$ simple closed curve $C \in \Omega$.

(2) Let $C(a) = \{z \in \mathbb{C} \mid |z-a| = 1\}$, positively oriented

Then $\int_{C(a)} \frac{1}{z-a}\,dz = 2\pi i$

$$\left(\int_{C(a)} \frac{1}{z-a}\,dz = \int_0^{2\pi} \frac{ie^{i\theta}}{(a+e^{i\theta})-a}\,d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i\right)$$

Now we consider integration of rational function.
Recall that a rational function $R(x)$ has the form

$$R(x) = \frac{p(x)}{q(x)}$$ (polynomials)

Example: 1) Compute $\int_{C(i)} \frac{z^3 - 2z^2 + 3z - 2}{z^2+1}\,dz$

Ans: Step 1: Long division.

$$
\begin{array}{r}
z - 2 \\
z^2+1 \overline{\smash{)}\, z^3 - 2z^2 + 3z - 2} \\
\underline{z^3 \qquad\quad + z \quad} \\
-2z^2 + 2z - 2 \\
\underline{-2z^2 \qquad\quad - 2} \\
2z
\end{array}
$$

$$\therefore \frac{z^3-2z^2+3z-2}{z^2+1} = z-2 + \frac{2z}{z^2+1}$$

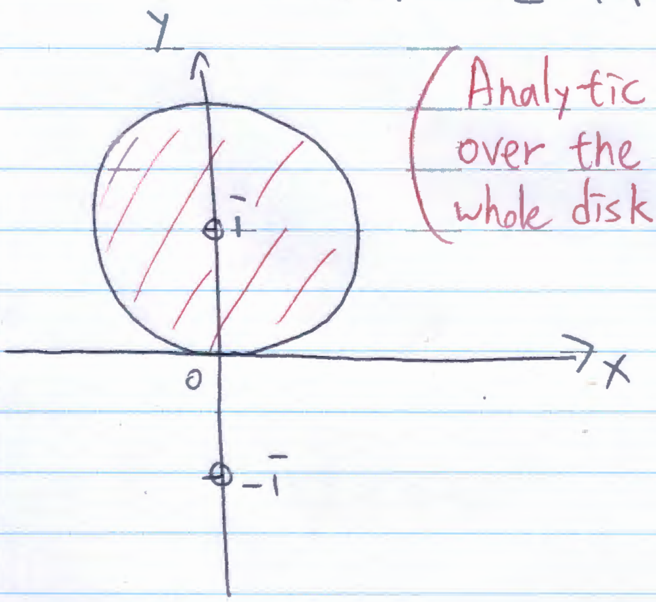
Step 2: Partial fraction.

let $\frac{2z}{z^2+1} = \frac{2z}{(z-i)(z+i)} = \frac{A}{z-i} + \frac{B}{z+i}$.

$$\Rightarrow \begin{cases} 2 = A+B \\ 0 = A-B \end{cases} \Rightarrow A = B = 1$$

Step 3: Apply Cauchy-Goursat theorem

$$\int_{C(i)} \frac{z^3-2z^2+3z-2}{z^2+1} = \int_{C(i)} (z-2)\,dz + \int_{C(i)} \frac{1}{z-i}\,dz$$



(Analytic over the whole disk)

$$+ \int_{C(i)} \frac{1}{z+i}\,dz$$

$$= 0 + 2\pi i + 0$$

$$= 2\pi i_{//}$$